Calculus One by Abdulaziz Al Ghannami, Quantq8.com

Chapter 1 Limits and Continuity

A set is a collection of elements.
A subset is a set within a set.

The Set of Natural Numbers

$N = \{1, 2, 3, \ldots\}$

The Set of Integers

$Z = \{\ldots, -1, 0, 1, 2, 3, \ldots\}$

The Set of Rational Numbers

$$Q = \left\{ \frac{a}{b} : \text{where } a \text{ is in } Z \text{ and } b \text{ is in } Z \text{ and } b \neq 0 \right\}$$

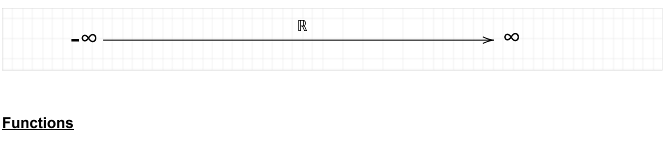
$$eg.\ \frac{1}{2}, \frac{3}{4} \text{ are in } Q$$

$$eg.\ \frac{\sqrt{2}}{2}, \frac{\pi}{4} \text{ are not in } Q$$

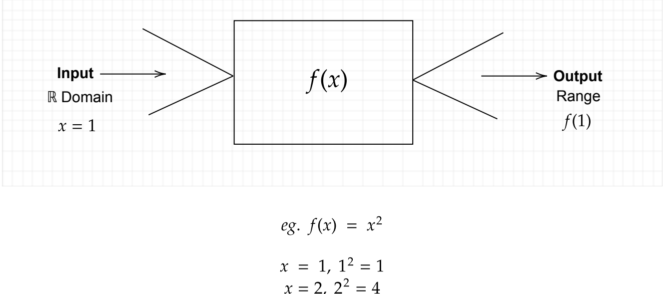
The numbers not in Q are called irrational numbers.

The Real Line

Is the set of real numbers. It is a line whose points are either in Q or are irrational numbers.



Functions



$$eg.\ f(x) = x^2$$

$$x = 1,\ 1^2 = 1$$
$$x = 2,\ 2^2 = 4$$
$$x = -2,\ (-2)^2 = 4$$
$$y = \sqrt{2},\ (\sqrt{2})^2 = 2$$

$$etc.$$

For a function to be a real-valued function it must consist of the following:

1. A set D of real numbers called the Domain
2. A rule/mechanism that associates with every real number in D, exactly one, number y, in range R

y is denoted as $f(x)$, and called the value of the function of x.
R is the range, and is the set of all values $f(x) = y$ of the function.

Families of Functions

The Polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,$$
$$\text{where } a_n \in \mathbb{R},\ n \in \mathbb{N}$$

$$\text{if } a_n \neq 0, \text{ then this is a polynomial of degree } n, \text{ with coefficients } a_0, a_1, \ldots, a_n$$

Examples of polynomials.

$f(x) = 4$	*Constant function*
$f(x) = x - 1$	*Linear function*
$f(x) = 2x^2$	*Quadratic function*
$f(x) = x^2 + x + 1$	*Quadratic function*
$f(x) = 5x^3 + 2x^2 - 7x + 4$	*Cubic function*

Power Functions

$f(x) = x^a,\ a \in \mathbb{R}$

It is important to be able to distinguish power functions from polynomial function. We can do so by asking a series of questions which will enlighten us on these differences:

is $f(x) = x^2$ a polynomial? YES

is $f(x) = x^{\frac{1}{2}}$ a polynomial? NO

$g(x) = x^2$	*polynomial & power function*
$h(x) = x^3 + 1$	*only polynomial function*
$f(x) = 3x^2$	*only polynomial function*

Rational Functions

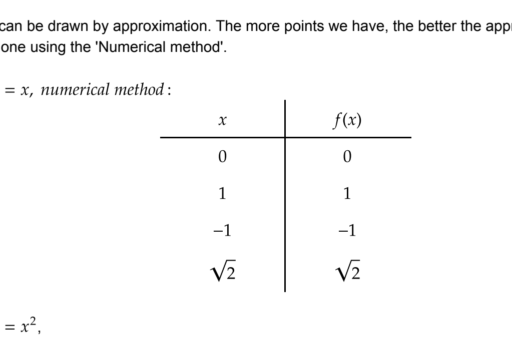
$f(x) = \frac{P(x)}{Q(x)}$, where both $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$

Examples of rational functions include;

$f(x) = \frac{2x + 3}{x^2 - 3x + 2}$	*rational function*
$g(x) = \frac{x^2 - 4}{2x}$	*rational function*
$h(x) = \frac{x}{1}$	*rational function*
$i(x) = \frac{x + 1}{x}$	*only composite function*

Graphs

The graph of a function f is the set of all pairs $(x,\ f(x))$, we draw it in the x-y plane.

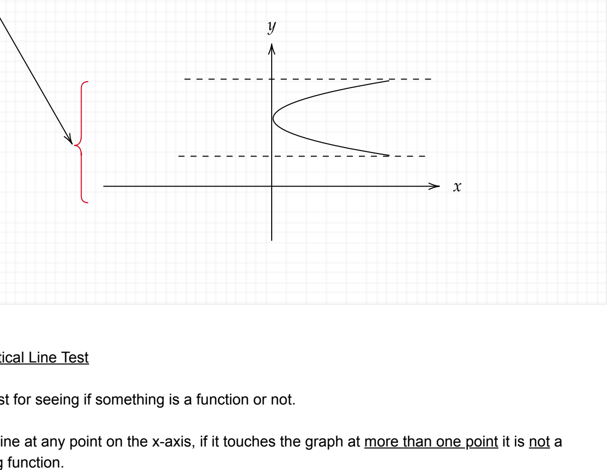


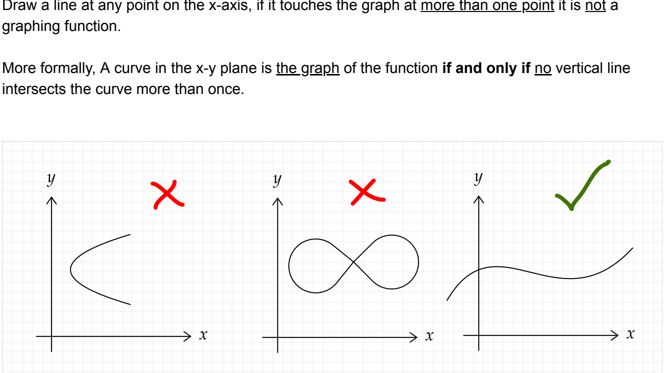
Graphs can be drawn by approximation. The more points we have, the better the approximation. This is done using the "Numerical method".

eg. $f(x) = x$, *numerical method* :

x	$f(x)$
0	0
1	1
−1	−1
$\sqrt{2}$	$\sqrt{2}$

eg. $f(x) = x^2$.



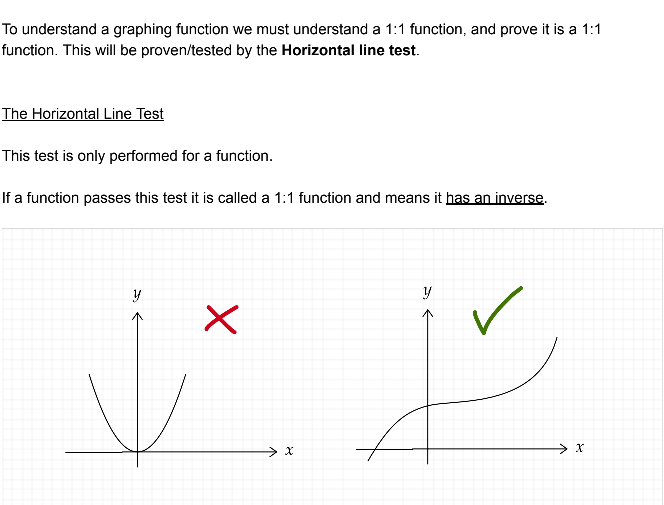


The Vertical Line Test

It is a test for seeing if something is a function or not.

Draw a line at any point on the x-axis, if it touches the graph at more than one point it is not a graphing function.

More formally, A curve in the x-y plane is the graph of the function **if and only if** no vertical line intersects the curve more than once.

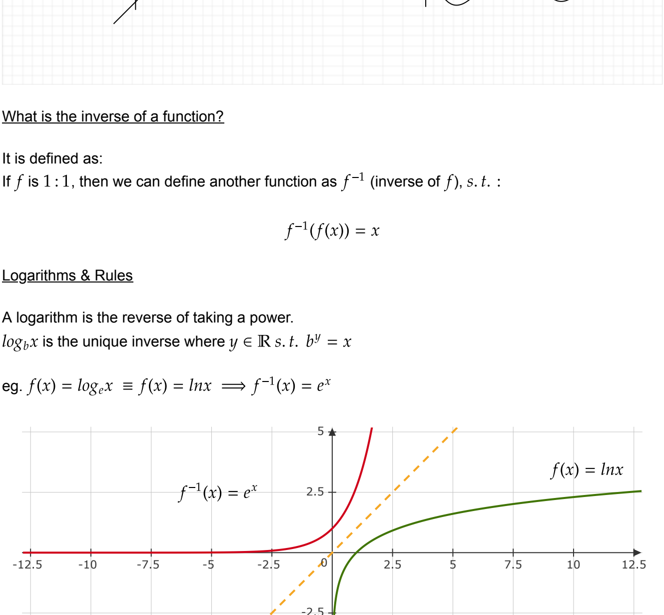


To understand a graphing function we must understand a 1:1 function, and prove it is a 1:1 function. This will be proven/tested by the **Horizontal line test**.

The Horizontal Line Test

This test is only performed for a function.

If a function passes this test it is called a 1:1 function and means it has an inverse.



What is the inverse of a function?

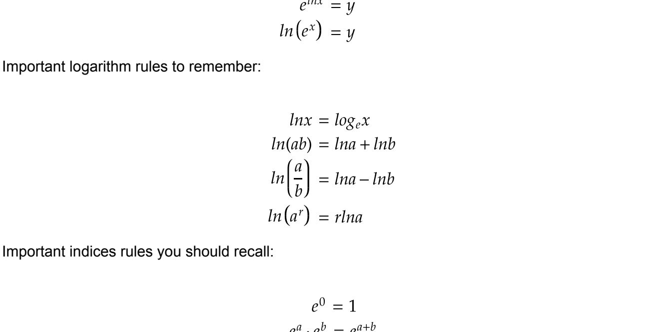
It is defined as:
If f is $1:1$, then we can define another function as f^{-1} (inverse of f), $s.t.$:

$$f^{-1}(f(x)) = x$$

Logarithms & Rules

A logarithm is the reverse of taking a power.
log_bx is the unique inverse where $y \in \mathbb{R}\ s.t.\ b^y = x$

eg. $f(x) = log_ex \equiv f(x) = lnx \implies f^{-1}(x) = e^x$



Since $f(x)$ is 1:1 it has an inverse $f^{-1}(x)$. In the graph this is evident by either curves reflections in the $y = x$ line.

Important logarithmic results to remember:

$$ln(0) = DNE$$
$$ln(1) = 0$$
$$e^{lnx} = y$$
$$ln(e^x) = y$$

Important logarithm rules to remember:

$$lnx = log_ex$$
$$ln(ab) = lna + lnb$$
$$ln\left(\frac{a}{b}\right) = lna - lnb$$
$$ln(a^r) = rlna$$

Important indices rules you should recall:

$$e^0 = 1$$
$$e^a \cdot e^b = e^{a+b}$$
$$\frac{e^a}{e^b} = e^{a-b}$$
$$(e^a)^r = e^{ar}$$

Exponential Function

$f(x) = a^x,\ where\ a > 0$

$eg.\ 2^x,\ 3^x,\ \left(\frac{1}{2}\right)^x,\ e^x$

If $x = \frac{q}{p}$ in the simplest form, q,p $\in Z$

then, $a^{\frac{q}{p}} = \left(\sqrt[p]{a}\right)^q$

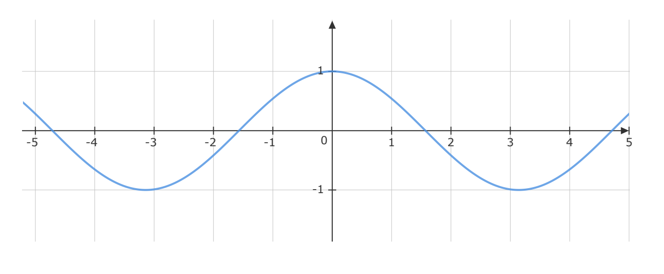
Trigonometric Functions

Sine

$$y = sinx$$

$$D_f = \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$$

$$R_f = [-1,\ 1]$$

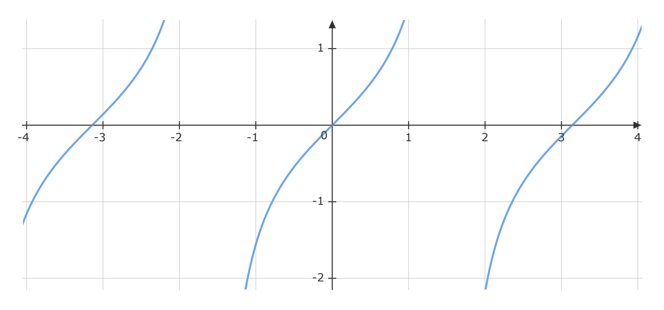


Cosine

$$y = cosx$$

$$D_f = [0,\ \pi]$$

$$R_f = [-1,\ 1]$$

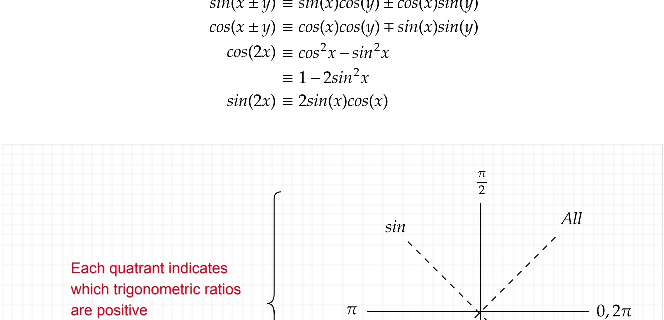


Tangent

$$y = tanx$$

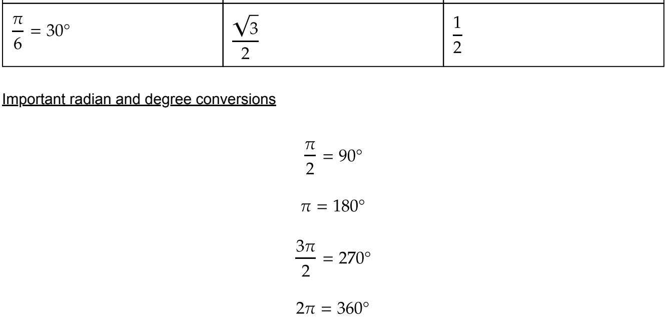
$$D_f = \left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$$

$$R_f = (-\infty,\ \infty)$$



Trigonometric Identities

$$sin^2x + cos^2x \equiv 1$$
$$1 + tan^2x \equiv sec^2x$$
$$1 + cot^2x \equiv cosec^2x$$
$$sin(x \pm y) \equiv sin(x)cos(y) \pm cos(x)sin(y)$$
$$cos(x \pm y) \equiv cos(x)cos(y) \mp sin(x)sin(y)$$
$$cos(2x) \equiv cos^2x - sin^2x$$
$$\equiv 1 - 2sin^2x$$
$$sin(2x) \equiv 2sin(x)cos(x)$$



θ	$cos\theta$	$sin\theta$
$\frac{\pi}{3} = 60°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4} = 45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{6} = 30°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

Important radian and degree conversions

$$\frac{\pi}{2} = 90°$$

$$\pi = 180°$$

$$\frac{3\pi}{2} = 270°$$

$$2\pi = 360°$$

Reciprocating the trigonometric functions

$$let,$$

$$f(x) = sinx$$
$$g(x) = cosx$$
$$h(x) = tanx$$

$$then,$$

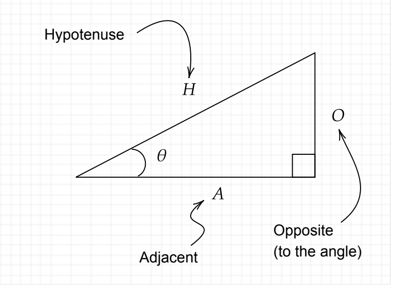
$f(x) = cosecx$
$h(x) = secx$
$l(x) = cotx$

Recall,

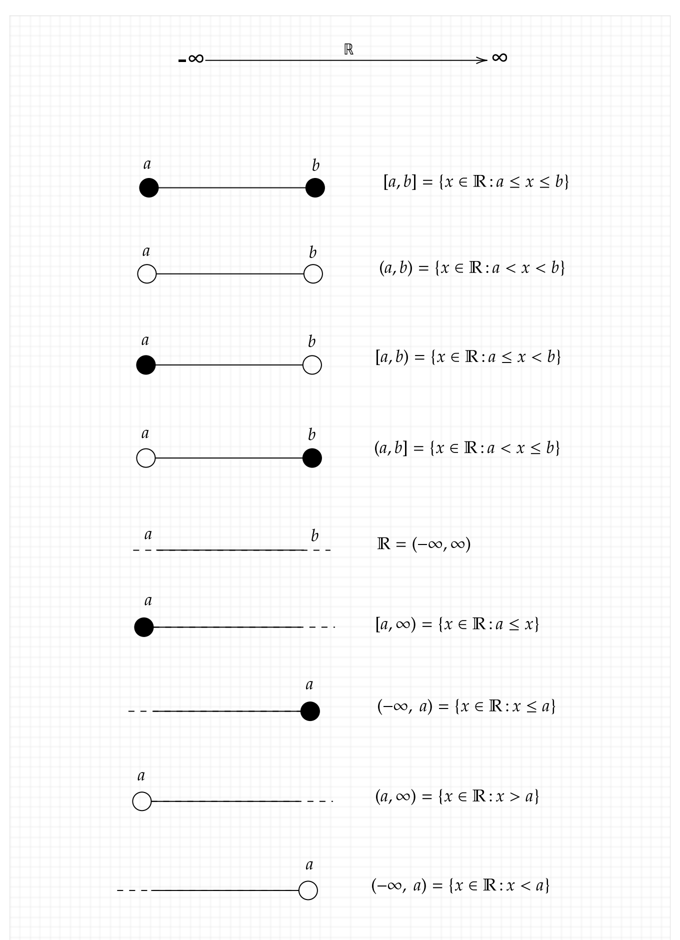
$sin\theta = \frac{O}{H}$

$cos\theta = \frac{A}{H}$

$tan\theta = \frac{O}{A}$



Intervals



Making new functions from old

If f and g are functions:

$(f+g)(x) = f(x) + g(x)$
$(f-g)(x) = f(x) - g(x)$
$(f \cdot g)(x) = f(x)g(x)$

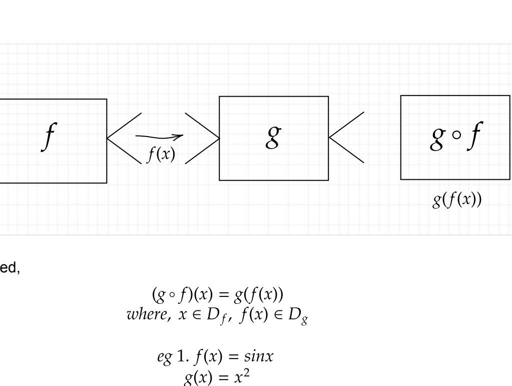
If $x \in D_f$ and also $x \in D_g$ where $g(x) \neq 0$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

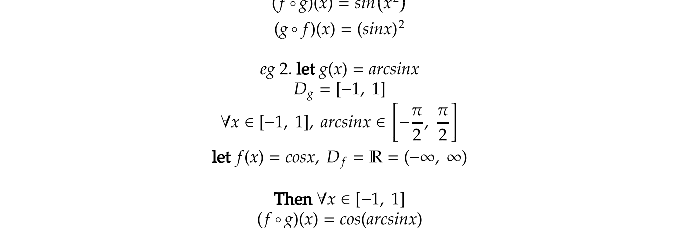
eg.

$f(x) = x + 1$
$g(x) = lnx$
$(f+g)(x) = x + 1 + lnx$
$(f-g)(x) = x + 1 - lnx$
$(f \cdot g)(x) = (x+1)lnx$
$\left(\frac{f}{g}\right)(x) = \frac{x+1}{lnx}$, for $x \in D_f \cap D_g$

Operations within functions to make new ones:



Composition of functions



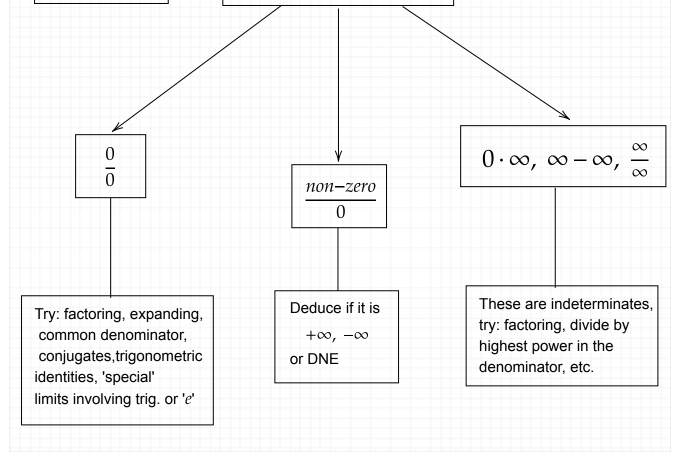
The above is called,

$$(g \circ f)(x) = g(f(x))$$
$$where,\ x \in D_f,\ f(x) \in D_g$$

$$eg\ 1.\ f(x) = sinx$$
$$g(x) = x^2$$
$$(f \circ g)(x) = sin(x^2)$$
$$(g \circ f)(x) = (sinx)^2$$

$$eg\ 2.\ let\ g(x) = arcsinx$$
$$D_g = [-1, 1]$$
$$\forall x \in [-1,1],\ arcsinx \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$let\ f(x) = cos,\ D_f = \mathbb{R} = (-\infty, \infty)$$
$$Then\ \forall x \in [-1,1]$$
$$(f \circ g) = cos(arcsinx)$$

Calculating Limits (Algebra)



In general:

If $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$

then, $\lim_{x \to x_0} f(x) = f(x_0)$

$\Longrightarrow a_0 + a_1x + a_2x^2 + ... + a_nx^n$

eg. $\lim_{x \to 1} \frac{2x+1}{x^2-1} = \frac{2 \cdot 1 + 1}{1^2 - 1} = \frac{3}{0} \Longrightarrow DNE$

$\therefore\ we\ write\ in\ this\ case,$

$\lim_{x \to 1^+} \frac{2x+1}{x^2-1} = \infty \Longrightarrow limit\ from\ the\ right$

$\lim_{x \to 1^-} \frac{2x+1}{x^2-1} = -\infty \Longrightarrow limit\ from\ the\ left$

$N.B.\ for\ \frac{0}{0} \Longrightarrow cannot\ conclude$

The Algebra of limits

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$

$$f(x) = ax^2 + bx + c$$
$$D = b^2 - 4ac$$

$$(D\ is\ the\ discriminant)$$

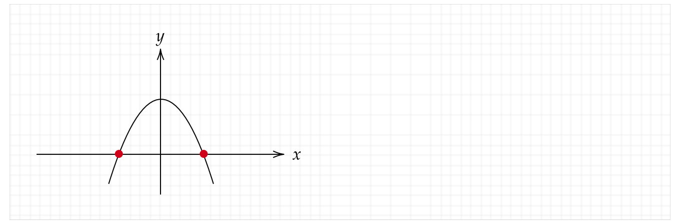
$$x_1 = \frac{-b + \sqrt{D}}{2a}$$

$$x_2 = \frac{-b - \sqrt{D}}{2a}$$

$$If$$

$D > 0$	$2\ Distinct\ real\ roots$
$D = 0$	$Double\ real\ roots$
$D < 0$	$No\ real\ roots$

Real roots touch the x-axis where $y = 0$



Limits of Rational Functions

$g(x) = \frac{P(x)}{Q(x)},\ where\ P(x)\ \&\ Q(x)\ are\ polynomials$

Case 1 $\lim_{x \to x_0} g(x) = \frac{P(x_0)}{Q(x_0)} = number\ or\ \left(\frac{0}{number} = 0\right)$

Case 2 $\lim_{x \to x_0} g(x) = \frac{P(x_0)}{Q(x_0)} = \frac{number\ (\neq 0)}{0} = DNE$

Case 3 $\lim_{x \to x_0} g(x) = \frac{P(x_0)}{Q(x_0)} = \frac{0}{0} = cannot\ conclude,\ must\ find\ another\ method\ to\ evaluate$

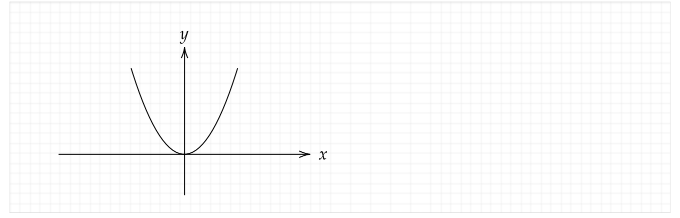
Side note

$A^3 + B^3 = \left(A^2 - AB + B^2\right)$
$A^3 - B^3 = (A - B)\left(A^2 + AB + B^2\right)$
$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
$(x - y)(x + y) = x^2 - y^2$

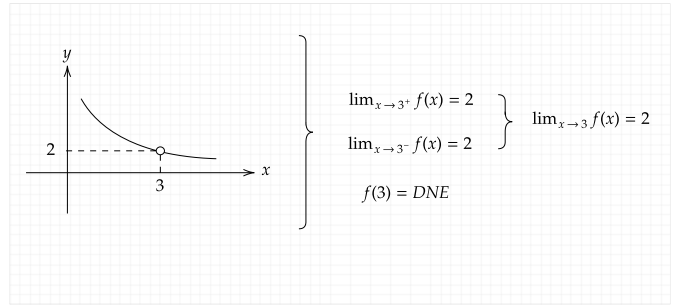
A function can be expressed in two ways:

Algebraically:
$eg.\ f(x) = x^2$

Geometrically:



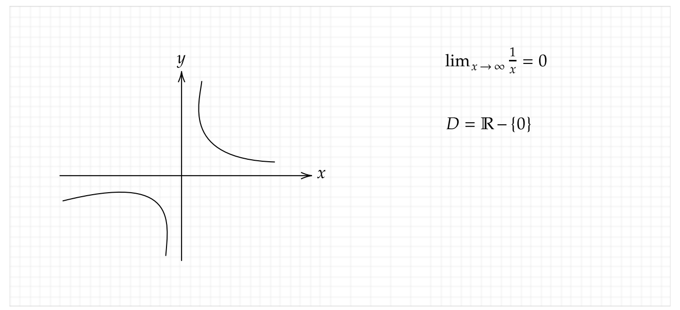
eg. of limits



When we talk about continuity we are talking specifically about the domain.

If a value isn't in the domain, continuity does not apply.
∴ there is a limit at 2, but no continuity at 2.

eg. of limits to infinity



You cannot conclude what a limit is in these two conditions:

$\frac{0}{0} = cannot\ conclude$

$\frac{\infty}{\infty} = cannot\ conclude$

$eg.$

$\lim_{x \to \infty} \frac{2}{x^2} = 0$
$\lim_{x \to -\infty} \frac{2}{x^2} = 0$

We can conclude that this is always true:

$\lim_{x \to \pm\infty} \frac{r}{x^n} = 0,\ r \in \mathbb{R}$

Examples of solving an inderminate form:

Ex.1
$\lim_{x \to \infty} \frac{2x+3}{5x-1} = \frac{\infty}{\infty} = cannot\ conclude\ (because\ it\ is\ of\ indeterminate\ form)$

When you have a polynomial over a polynomial (rational function) divide by x^n (multiply by $\frac{1}{x^n}$), n being the largest power (in the rational function).

$\lim_{x \to \infty} \frac{2x + 3 \cdot \frac{1}{x}}{5x - 1 \cdot \frac{1}{x}}$

$\lim_{x \to \infty} \frac{2 + \frac{3}{x}}{5 - \frac{1}{x}} = \frac{2+0}{5+0} = \frac{2}{5}$

Ex.2
$\lim_{x \to \infty} \sqrt{x^2 + 3x} - x = \infty - \infty = cannot\ conclude$

In this case we rationalize it, and make it a ratio of the form $\frac{a}{b}$

$\lim_{x \to \infty} \frac{\sqrt{x^2+3x} - x}{1}$

$\lim_{x \to \infty} \frac{\sqrt{x^2+3x} - x}{1} \cdot \frac{\sqrt{x^2+3x} + x}{\sqrt{x^2+3x} + x}$

$\lim_{x \to \infty} \frac{\left(x^2+3x\right) - x\sqrt{x^2+3x} + x\sqrt{x^2+3x} - x^2}{\sqrt{x^2+3x} + x} = \lim_{x \to \infty} \frac{\left(x^2+3x\right) - x^2}{\sqrt{x^2+3x} + x} = \lim_{x \to \infty} \frac{3x}{\sqrt{x^2+3x} + x}$

$\lim_{x \to \infty} \frac{3x}{\sqrt{x^2+3x} + x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$

recall,

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$
$$\frac{1}{\sqrt{x^2}} = \frac{1}{x}$$

so,

$\lim_{x \to \infty} \frac{3}{\sqrt{1 + \frac{3}{x}} + 1} = \frac{3}{\sqrt{1 + \frac{3}{\infty}} + 1} = \frac{3}{2}$

If it is a polynomial then consider its biggest power.
RULE:
$\lim_{x \to \pm\infty} \left(a_nx^n + ... + a_2x^2 + a_1x + a_0\right)$

$= \lim_{x \to \pm\infty} a_nx^n,\ gain\ either\ \infty\ or\ -\infty\ depending\ on\ n\ and\ a_n$

A few limits to remember

1- $\lim_{x \to \pm\infty} sinx = \lim_{x \to \pm\infty} cosx = DNE$

2- $\lim_{x \to \infty} e^x = \lim_{x \to \infty} lnx = \infty$

3- $\lim_{x \to -\infty} e^x = 0$ for $\lim_{x \to -\infty} lnx \Longrightarrow nonesensical$

4- $\lim_{x \to 0^+} lnx = -\infty$

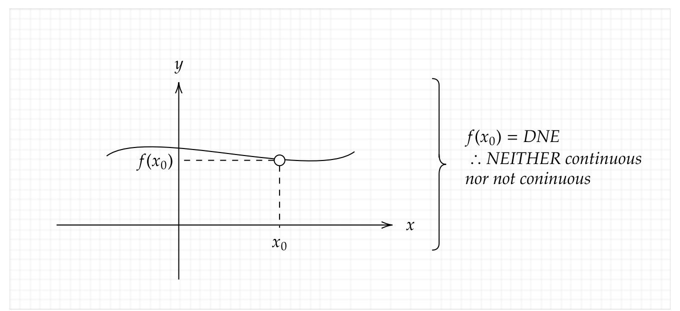
5- $\lim_{x \to \infty} arctanx = \frac{\pi}{2}$ & $\lim_{x \to -\infty} arctanx = -\frac{\pi}{2}$

Definition of Continuity

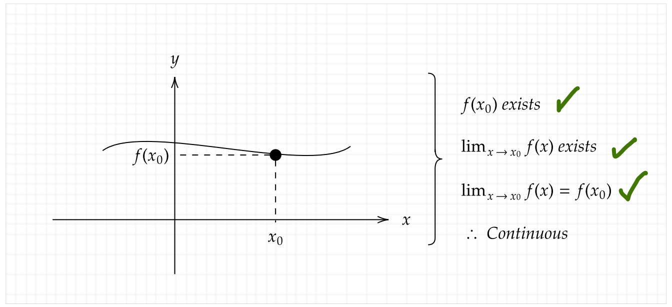
A function $f(x)$ is said to be continuous at $x = x_0$, if:

$$1 -\ f(x)\ is\ defined\ as\ x = x_0;\ ie.\ f(x_0)\ exists$$
$$2 - \lim_{x \to x_0} f(x)\ exists$$
$$3 - \lim_{x \to x_0} f(x) = f(x_0)$$

Eg 1.

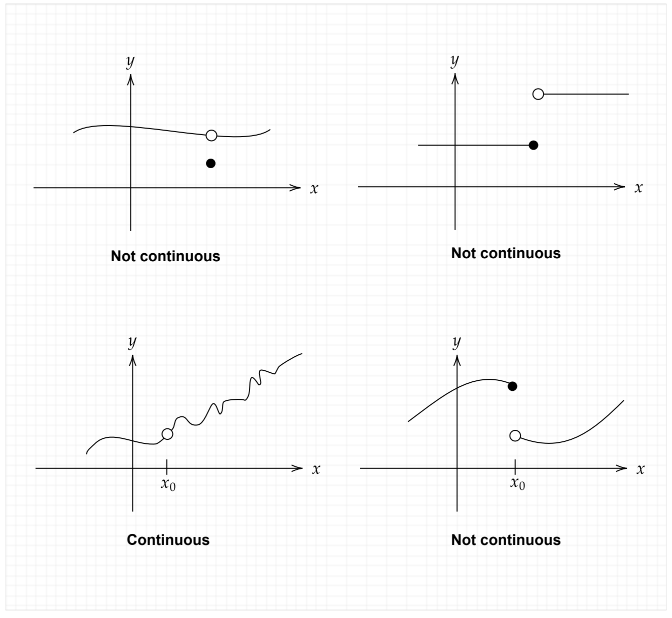


Eg 2.



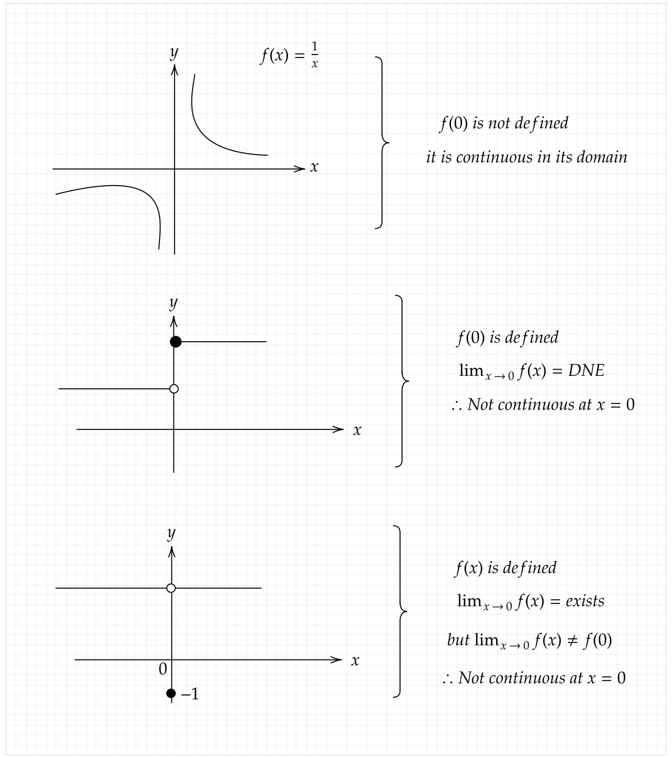
A function f is continuous if it is continuois for each $x_0 \in D_f$.

Visual examples of continuous and not continuous functions:

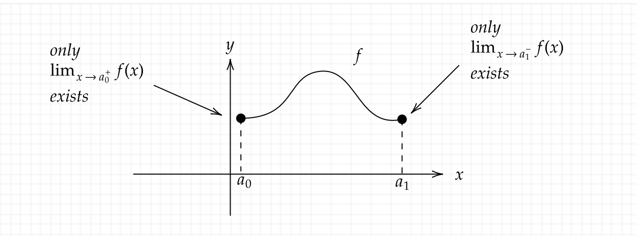


A **differentiable function** is continuous at all points in its domain.
A **continuous function** does not need to be differentiable.

eg.



A Note on Endpoints:



If $x = a_1$ is an endpoint for the domain of $f(x)$ then,

$$\lim_{x \to x_0} f(x)$$

(in the definition) is replaced by the appropriate left or right limit.

N.b '"limit from the left" is denoted by a '+' and "limit from the right" is denoted by a '-'

Recall the identity:

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ also remember that $|of\ a\ function|$ is continuous everywhere

Defintion of Limits

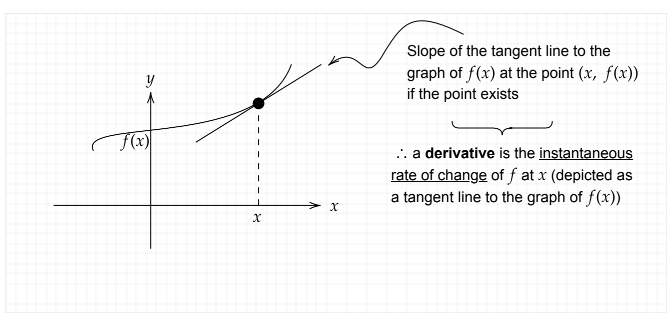
$$\begin{gathered}\text{Let } f(x) \text{ be a function and } x_0 \in \mathbb{R}, \\ s.t.\ x_0 \in D_f \text{ or } x_0 \notin D_f \\ \text{we say } \lim_{x \to x_0} f(x) = L \\ \text{If } f(x) \text{ can be made arbitrarily close to } L \text{ by showing } x \text{ sufficiently} \\ \text{close to (but not equal to) } x_0\end{gathered}$$

Recall,
Definition of Continuity

A function $f(x)$ is said to be continuous at $x = x_0$, if:

$$\begin{gathered}1- f(x) \text{ is defined as } x = x_0;\ ie.\ f(x_0) \text{ exists} \\ 2- \lim_{x \to x_0} f(x) \text{ exists} \\ 3- \lim_{x \to x_0} f(x) = f(x_0) \text{ or } \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) \text{ or Endpoints (replacing def)}\end{gathered}$$

Definition of Derivative



The derivative is defined as a limit as such (this is referred to as 'the definition of a derivative'):

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

eg. Calculate the derivative of $f(x) = \sqrt{x},\ x \geqslant$ using the definition,

$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$

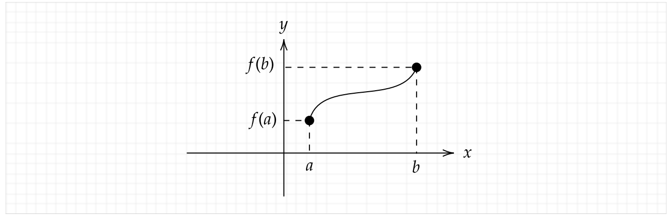
$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$

$= \lim_{h \to 0} \frac{1}{\sqrt{x} \cdot \sqrt{x}}$

$= \lim_{h \to 0} \frac{1}{\sqrt{2x}},\ D_f(0, \infty)$

The Intermediate Theory

The General Case

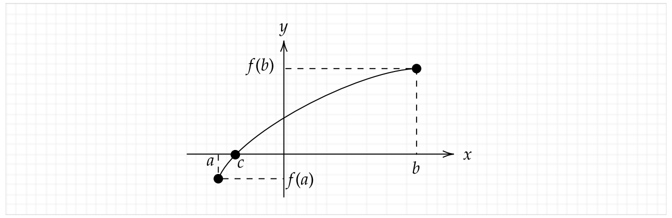


If

- $f(x)$ is continuous on $[a, b]$
- $f(a) \neq f(b)$

Then for all y_0 inbetween $f(a)$ & $f(b)$, there exists $x_0 \in (a, b)$ such that $f(x_0) = y_0$

The Special Case



If

- $f(x)$ is continuous on $[a, b]$
- Have opposite signs

Then there exists at least one c such that $c \in (a, b),\ f(c) = 0$

eg. $x^3 + 3x - 2 = 0$, this equation is hard to solve for,

However there is a solution in $(0, 1)$
For this, consider $f(x) = x^3 + 3x - 2$ on $[0, 1]$
$f(0) = -2 < 0$
$f(1) = -2 > 0$
IVT say $\exists\, c \in (0, 1)$ such that $f(c) = 0,\ c^3 + 3c - 2 = 0$

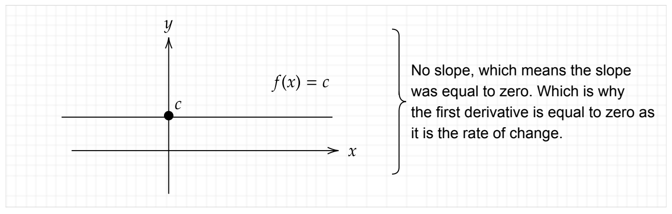
Chapter 2 Differentiation

Differentiation techniques

Imagine having to find out the derivative of a function by using the definition. Luckily we have a method to find the derivative of any function in general:

$$\begin{gathered}\text{for a function } f(x) = x^n, \\ \text{the derivative } f'(x) = nx^{n-1}\end{gathered}$$

N.b. The derivative of any constant function is zero.
$f(x) = c,\ c \in \mathbb{R}$
$f'(x) = c' = 0$
eg. $g(x) = 3,\ g'(x) = 0$



Differentiation rules:

- For any constant c,
  $(cf(x))' = cf'(x))$
- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- Product rule:
  $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- Quotient rule:
  $g(x) \neq 0,\ \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$
- Chain rule: derivative of a compositve function
  Recall, $(f \circ g)(x) = f(g(x))$
  $[f(g(x))]' = f'(g(x)) \cdot g'(x)$ *Newtons notation*
  $\frac{df(g(x))}{dy} \Longrightarrow \text{let } y = f(u)\ \&\ u = g(x)\ \therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ *Liebinz notation*

Linear approximation & differentials

Linear aprroximation for $f(x)$ at a
$f(x) \approx f(a) + f'(a)(x - a)$
if we have $f(x)$ and x changes from x_0 to x_1 $\longrightarrow \Delta x \longrightarrow (x_1 - {}_0)$
$\therefore f(x_1) \approx f(x_{0)}) + f'(x_0)$

Linearization of f at a
$L(x) = f(a) + f'(a)(x - a)$
$\Delta y = f(x + \Delta x) - f(x)$

Derivatives of General Exponential Functions

$f(x)$	$f'(x)$
a^x	$a^x ln(a)$
$a^{f(x)}$	$a^{f(x)} \cdot f'(x) \cdot ln(a)$

Derivatives of Logarithmic Functions

$f(x)$	$f'(x)$
lnx	$\frac{1}{x}$
$log_a(x)$	$\frac{\frac{1}{x}}{lna} = \frac{1}{x \cdot lna}$
$log_a(f(x))$	$\frac{f'(x)}{lna \cdot f(x)}$

Recall this rule,
$log_a x = \frac{lnx}{lna}$

We can use logarithms to simply differentiated complicated functions.
if $y = y_1 \pm y_2 \pm y_3$
then $y' = y_1' \pm y_2' \pm y_3'$

Derivative of the inverse trigonometric functions

$f(u)$	$f'(u)$
$sin^{-1}(u)$	$\frac{1}{\sqrt{1-u^2}} \cdot u'$
$cos^{-1}(u)$	$\frac{-1}{\sqrt{1-u^2}} \cdot u'$
$tan^{-1}(u)$	$\frac{1}{u^2+1} \cdot u'$
$cot^{-1}(u)$	$\frac{-1}{u^2+1} \cdot u'$
$sec^{-1}(u)$	$\frac{1}{u\sqrt{u^2-1}} \cdot u'$
$csc^{-1}(u)$	$\frac{-1}{u\sqrt{u^2-1}} \cdot u'$

Minimum & Maximum values

It is a critical number in a function f is a number c in the domain of f and that $f'(c) = 0$ & $f'(c) = DNE$

The absolute maximum and minimum (extrema values) in the interval $[a, b]$:

- find the critical number c in the interval (a, b)
- find $f(c)$
- find $f(b)$ & $f(a)$
- The smallest value is the minimum
- The biggest value is the maximum

Rolle's Theorem

Let f be a function that satisfies the following:

1. f is continuous on $[a, b]$
2. f is differentiable on (a, b)
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$

Things we want to show from an equation for $f(x)$, that it has:

- Exaclty one solution = at most one solution a unique solution
- Find a solution using IVT theorem
- Proof by contradictionn

Rolle's Theorem Application:

eg. prove that $x^3 + x - 3 = 0$ has exactly one real root.

The steps intutional steps we will follow:

- Show that it has a root (Step 1)
- then, Show that that is the only real root (Step 2)

Step 1- It has a root, by IVT:

$f(x) = x^3 + x - 3$
$f(0) = 0 + 0 - 3 = -3 < 0$
$f(2) = 2^3 + 2 - 3 = 7 > 0$

it is a continuous polynomial, the results above show that it transition from negative to positive therefore there will be atleast one real root in the interval $(-3, 7)$

Step2- It has only one real root

let it have 2 real roots, $x = a,\ x = b$
$f(a) = 0,\ f(b) = 0$ real root
$\therefore f(a) = f(b) = 0$

So to satisfy Rolle's Theorem: Continuous & Differentiable & $f(a) = f(b)$
$\therefore f'(c) = 0$
$f(x) = x^3 + x - 3$
$f'(x) = 3x^2 + 1 \quad > 1$ (always positive)

Since $f'(x) \neq 0$, our assumption is wrong given the equation has exactly one real root.

Proof by contradiction calculation:

1. Show at least one real root exists using IVT
2. Assume two or more roots exist
3. Assuming in step 2 and another known fact (MVT/Rolle's Theorem) show that something else must occur
4. Show that the "something else" cannot occur, simplifying something, so the hypothesis is wrong.

Keep in mind these fact:

- if c is a critical point for $f(x)$ then $f'(c) = 0$
- if a is an inflection point for $f(x)$ then $f''(a) = 0$

Mean Value Theorem (MVT)

Let f be a function that satisfies the following hypothesis:

- f is continuous on $[a, b]$
- f is differentiable on (a, b)

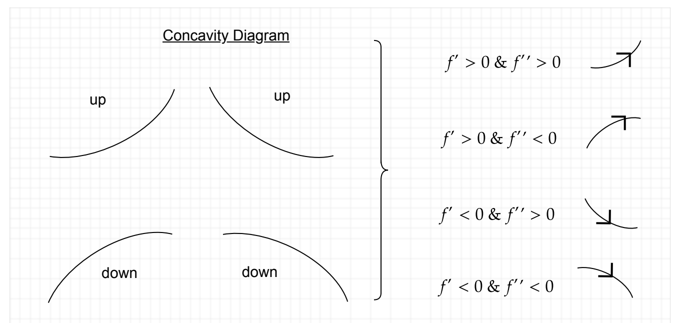
Then there is a number $c \in (a, b)$
such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Sketching Graphs

- **Relative/local maximum:** f changes from $\nearrow$ to $\searrow$
  f is $\nearrow$ at an interval if $f'(x) > 0$
- **Relative/local maximum:** f changes from $\searrow$ to $\nearrow$
  f is $\searrow$ at an interval if $f'(x) < 0$
- **Critical points** are points where the derivative $f'(x)$ is either 0 or undefined (DNE)
- **Extrema** (absolute max and min) occur at critical points, but not every critical poinnt is an extrema

To determine Extrema we must do 2 things:

1. find the critical points:
   (a) Compute $f'(x)$
   (b) Equate $f'(x)$ to 0
   (c) Solve $f'(x) = 0$, if solutions are 0 or DNE then a critical point has been found
2. "Test" each critical point (found in step 1) to determine if it is a local max, local min or neither



The 6 steps for how to sketch a graph

1. Find all the points at which the behaviour of the graph could change
2. Disregard points not in the D_f
3. Points in the D_f if they exist are critical points, if sols:
   $f'(x) = 0 \ or \ f'(x) = DNE$ (for finding extrema; absolute max & min)
4. Inflection points: $f''(x) = 0 \ or \ f''(x) = DNE$
5. Using the concavity diagram and the data just collected, piece all the curves above, being careful to indicate any horizontal $(f'(x) = 0)$ or vertical $\left(f'(x) = DNE\right)$ asymptotes
6. When necessary, add intercepts (x=0 y-intercept & y=0 x-intercept) and horizontal asympototes

Hyperbolic functions

$$\begin{aligned}
sinhx &= \frac{1}{2}\left(e^x - e^{-x}\right)\\
coshx &= \frac{1}{2}\left(e^x + e^{-x}\right)\\
tanhx &= \frac{sinhx}{coshx}\\
cothx &= \frac{coshx}{sinhx}\\
sechx &= \frac{1}{coshx}\\
cosechx &= \frac{1}{sinhx}\\
(sinhx)' &= coshx\\
(coshx)' &= sinhx\\
(tanhx)' &= sech^2x\\
(cothx)' &= -csch^2x\\
(sechx)' &= -sechx \cdot tanhx\\
(cschx)' &= -cschx \cdot cothx\\
sinh^{-1}(x) &= ln\left(x + \sqrt{x^2+1}\right)\\
cosh^{-1}(x) &= ln\left(x + \sqrt{x^2-1}\right)\\
tanh^{-1}(x) &= \frac{1}{2} ln\left(\frac{1+x}{1-x}\right)
\end{aligned}$$

Optimization

The Steps needed to solve an optimization problem:

- Looking for the largest or smallest value (Extrema, absolute max & min) of a function subject to some kind of **constraint**
- The **constraint** will be some condition. The condition will be described by an equation.
- Identify the quantity to be optimized and the constraint (must be true regardless of the solution)
- Always will have a minimum of 2 functions:
  - a minimization/maximization function
  - constraint has a fixed value
- Rearrange the contraint and substitute it in the function
- Differentiate and obtain the first derivative to find the absolute extrema
- Use the second derivative to confrim if it is a max or min

Chapter 3 Integration

Antiderivatives/indefinite integrals, definite integrals/fundemental theorem of calculus

Given a function $f(x)$, an anti derivative for $f(x)$ is another function $F(x)$ which satisfies the following $F'(x) = f(x)$

$f(x)$ —— differentiation ——> $f'(x)$

$f(x)$ —— anti-differentiation ——> $F'(x), \ F'(x) = f(x)$

All anti-derivatives are in the family of integrals

Notation

$\int f(x) \cdot dx$, it is the indefinite integral of $f(x)$, it is the collection of all anti-derivatives of $f(x)$, $= F(x) + c$, c is the constant of integration;

where $F(x)$ is any function satisfying, $F(x) = f(x), \ c \in \mathbb{R}$

Rule,

$$\int x^n \cdot dx = \frac{1}{n-1} \cdot x^{n+1} + c, \ n \neq -1$$

$$exception: \ x^{-1} = \frac{1}{x}, \ \int \frac{1}{x} \cdot dx = ln|x| + c$$

A table of Integrals that you should know

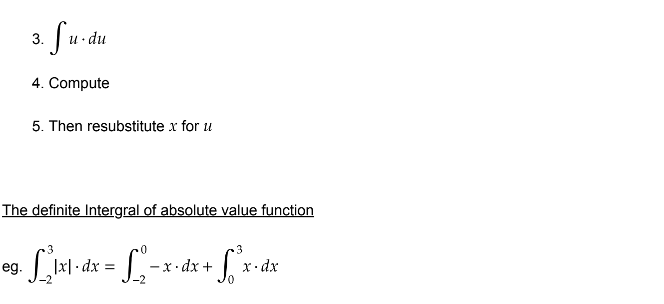
$$\begin{aligned}
\int 1 \cdot dx &= x + c\\
\int x^n \cdot dx &= \frac{x^{n+1}}{n+1} + c\\
\int \frac{1}{x} \cdot dx &= ln|x| + c\\
\int e^x \cdot dx &= e^x + c\\
\int b^x \cdot dx &= \frac{b^x}{lnb} + c\\
\int sinx \cdot dx &= -cosx + c\\
\int cosx \cdot dx &= sinx + c\\
\int sec^2x \cdot dx &= tanx + c\\
\int csc^2x \cdot dx &= -cotx + c\\
\int secxtanx \cdot dx &= secx + c\\
\int cscxcotx \cdot dx &= -cscx + c\\
\int tanx \cdot dx &= ln|secx| + c\\
\int cotx \cdot dx &= ln|sinx| + c\\
\int sinhx \cdot dx &= coshx + c\\
\int coshx \cdot dx &= sinhx + c\\
\int sech^2x \cdot dx &= tanhx + c\\
\int csch^2x \cdot dx &= -cothx + c\\
\int sechxtanhx \cdot dx &= -sechx + c\\
\int cschxcothx \cdot dx &= -csch + c\\
\int \frac{1}{\sqrt{1-x^2}} \cdot dx &= arcsinx + c\\
\int \frac{1}{1+x^2} \cdot dx &= arctanx + c\\
\int \frac{1}{x\sqrt{x^2-1}} \cdot dx &= arcsecx + c
\end{aligned}$$

Properties

1. $\int_a^b (f(x) \pm g(x)) \cdot dx = \int_a^b f(x) \cdot dx \pm \int_a^b g(x) \cdot dx$ *it is for larger sums also*
2. For any constany c,
   $\int_a^b cf(x) \cdot dx = c\int_a^b f(x) \cdot dx$
3. $\int_a^b f(x) \cdot dx = \int_a^c f(x) \cdot dx + \int_c^b f(x) \cdot dx, \ c \in \mathbb{R}$

Also, $\int_a^b f(x) \cdot dx = \int_c^b f(x) \cdot dx - \int_c^a f(x) \cdot dx$
$= \int_c^b f(x) \cdot dx + \int_a^c f(x) \cdot dx$

N.b is true even if $c \notin [a, b]$



Integration by substitution

The steps:

1. $(\square)^n$ let $\square = u$ usually choose the most complicated
2. $\frac{du}{dx} = \square'$ $du = \square' \cdot dx$
3. $\int u \cdot du$
4. Compute
5. Then resubstitute x for u

The definite Intergral of absolute value function

eg. $\int_{-2}^{3} |x| \cdot dx = \int_{-2}^{0} -x \cdot dx + \int_0^3 x \cdot dx$

where x becomes 0 is where we break up the function,
Why do we break it?
As there are 2 different linear functions for $x \in (-\infty, 0)$ & for $[0, \infty)$.

N.B. 1 in an odd function (odd number)
$f(-x) = -f(x)$
$\therefore \int_{-a}^{a} f(x) \cdot dx = 0$
eg. of odd functions: $sinx, \ tanx, \ x^3, \ x^5, \ x^7$, ...

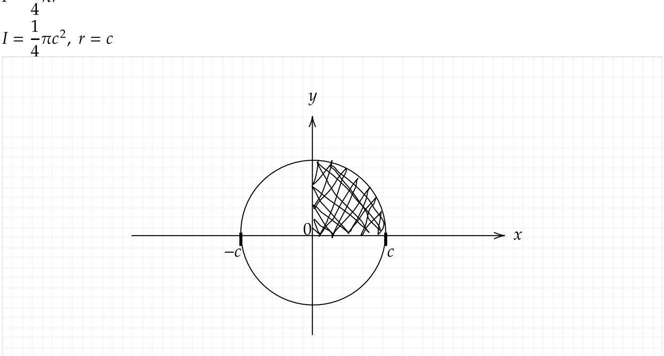
N.B. 2 in even functions
$f(-x) = f(x)$
$\therefore \int_{-a}^{a} f(x) \cdot dx = 2\int_0^a f(x) \cdot dx$
eg. of even functions: $cosx, \ x^2, \ x^4, \ x^6$, ...

- first let $f(x)$ equal the function then test if even or odd then use ' $\therefore$ ' deduction to solve the answer
- if the integral has $\int_{-a}^{a}$ always think of odd and even functions

Area of a circle

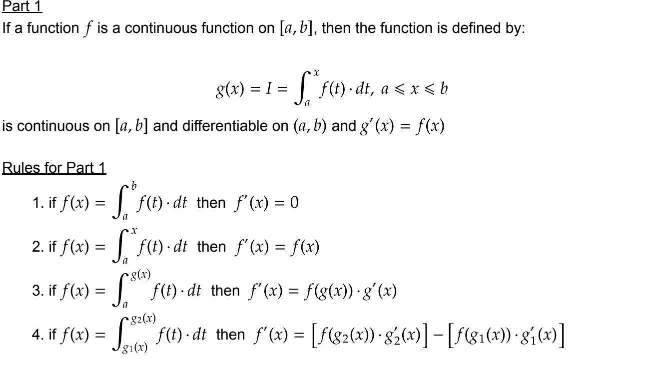
Form 1:
$I = \int_{-c}^{c} \sqrt{c^2 - x^2} \cdot dx$

Deduction:
$I = \frac{1}{2}\pi r^2$
$I = \frac{1}{2}\pi c^2, \ r = c$



Form 2:
$I = \int_0^c \sqrt{c^2 - x^2} \cdot dx$

Deduction:
$I = \frac{1}{4}\pi r^2$
$I = \frac{1}{4}\pi c^2, \ r = c$



So effectively to solve such integrals we can rely on the deductions instead of performing the lengthy integration.

The Fundamental Theorem of Calculus

Part 1
If a function f is a continuous function on $[a, b]$, then the function is defined by:

$$g(x) = I = \int_a^x f(t) \cdot dt, \ a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$

Rules for Part 1

1. if $f(x) = \int_a^b f(t) \cdot dt$ then $f'(x) = 0$
2. if $f(x) = \int_a^x f(t) \cdot dt$ then $f'(x) = f(x)$
3. if $f(x) = \int_a^{g(x)} f(t) \cdot dt$ then $f'(x) = f(g(x)) \cdot g'(x)$
4. if $f(x) = \int_{g_1(x)}^{g_2(x)} f(t) \cdot dt$ then $f'(x) = \left[f(g_2(x)) \cdot g_2'(x)\right] - \left[f(g_1(x)) \cdot g_1'(x)\right]$

How to prove that:

1. $f(x)$ is increasing, show $f'(x) > 0$
2. $f(x)$ is decreasing, show $f'(x) < 0$
3. $f(x)$ is constant, show $f'(x) = 0$

Find local extrema:
1. find critical points, by finding $f'(x)$
2. to get inflection point find $f''(x)$ and equat to 0

How to show if a curve is concave up or concave down:
Use the second derivative to show it,
if $f''(x) > 0$ concave upwards
if $f''(x) < 0$ concave downwards

Chapter 4 Applications of Differentiation and Integration

The First & Second Derivatives

The meaning of the first derivative

- The first derivative of the function $f(x)$ is $\left(f'(x) \ or \ \frac{df}{dx}\right)$ the slope of the tangent line to the function at the point x.
- It tells us whether a function is increasing or decreasing and by how much:
  - if $\frac{df}{dx}(p) > 0$, then $f(x)$ is an increasing function at $x = p$
  - if $\frac{df}{dx}(p) < 0$, then $f(x)$ is an decreasing function at $x = p$
  - if $\frac{df}{dx}(p) = 0$, then $x = p$ is called a critical point of $f(x)$, and we do not know the behaviour of $f(x)$ at $x = p$ (this is why we may then use the second derivative)

The meaning of the second derivative

- It is the derivative of the derivative of that function, $\left(f''(x) \ or \ \frac{d^2f}{dx^2}\right)$
- The second derivative tells us if the first derivative is increasing or decreasing
  - if $\frac{d^2f}{dx^2}(p) > 0$ at $x = p$, then $f(x)$ is concave up at $x = p$
  - if $\frac{d^2f}{dx^2}(p) < 0$ at $x = p$, then $f(x)$ is concave down at $x = p$
  - if $\frac{d^2f}{dx^2}(p) = 0$ at $x = p$, then we do not know anything new about the behaviour of$f(x)$ at $x = p$

Critical points & the second derivative test

When x is a critical point of the function $f(x)$, we do not learn anything new about the function at that point; it could be:

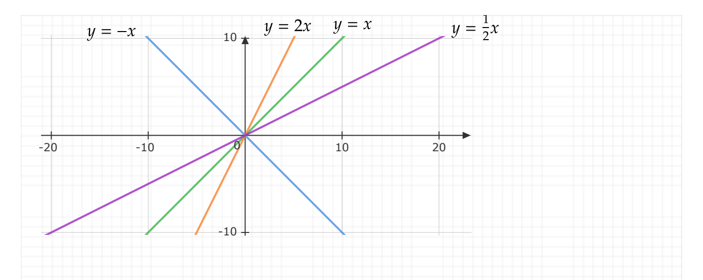
- Increasing
- Decreasing
- a local maximum
- a local minimum

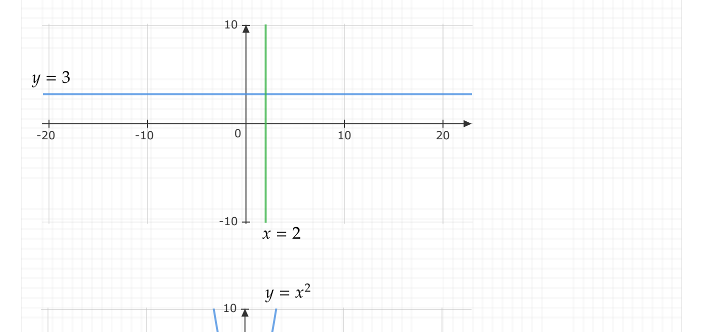
- if $\frac{df}{dx}(p) = 0$ & $\frac{d^2f}{dx^2}(p) > 0$, then $f(x)$ has a local minimum at $x = p$
- if $\frac{df}{dx}(p) = 0$ & $\frac{d^2f}{dx^2}(p) < 0$, then $f(x)$ has a local maximum at $x = p$
- if $\frac{df}{dx}(p) = 0$ & $\frac{d^2f}{dx^2}(p) = 0$, then we learn no new information about the behaviour of $f(x)$ at the point $x = p$

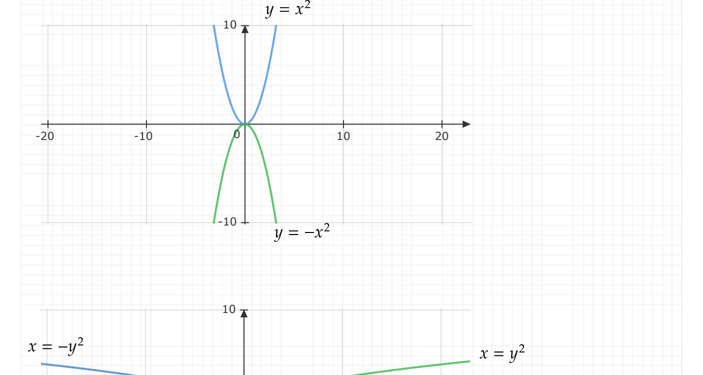
Inflection points

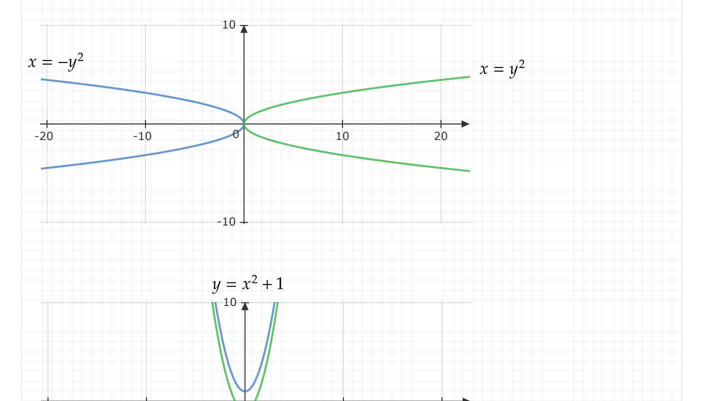
- A function $f(x)$ has an inflection point at x if the graph of the function goes from concave up to concave down or vice versa
- An innflection poinnt can onnly happen where the second derivative is 0
- It cannot tell us if the graph of a function has an inflection point; it can only tell us where it might have an inflection point

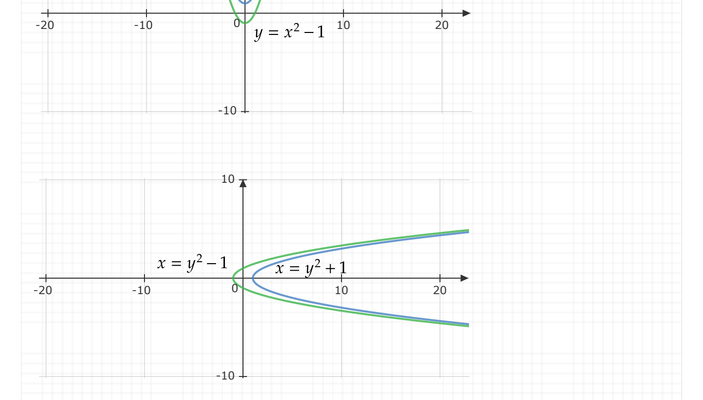
Graphs

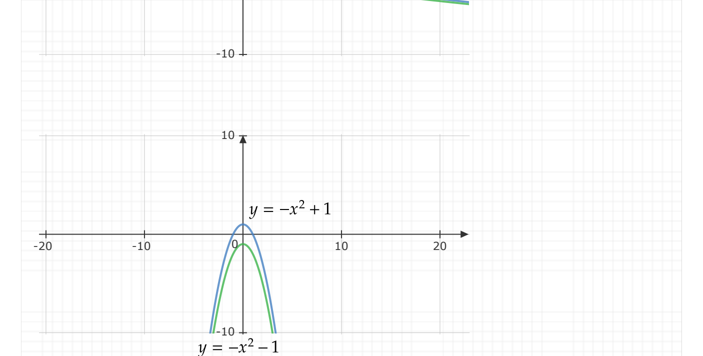


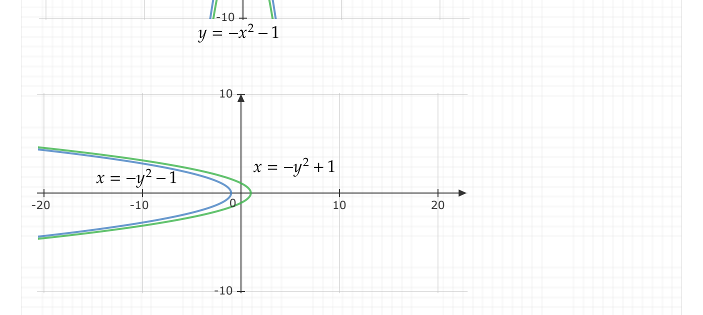


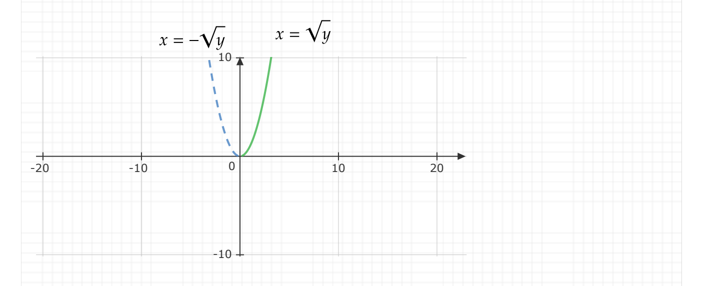


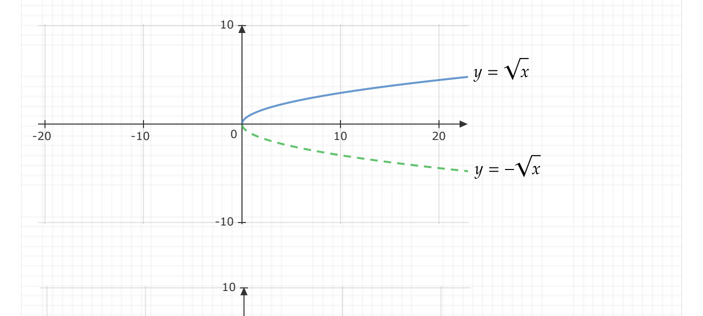


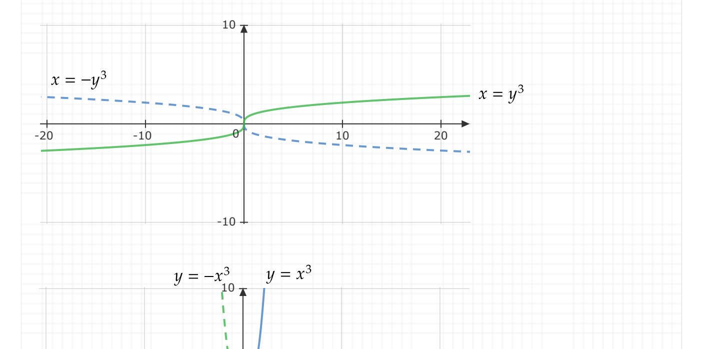


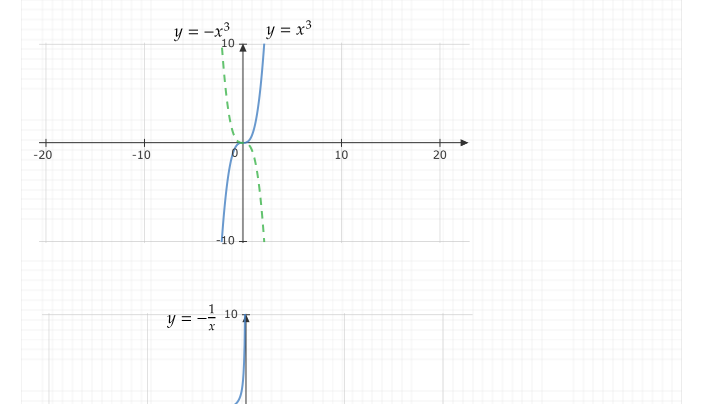


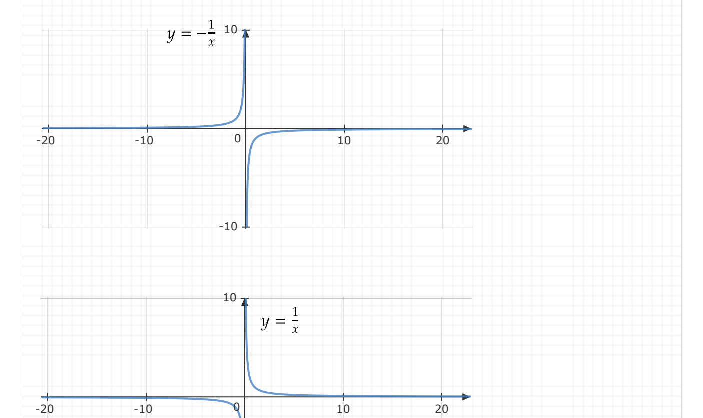


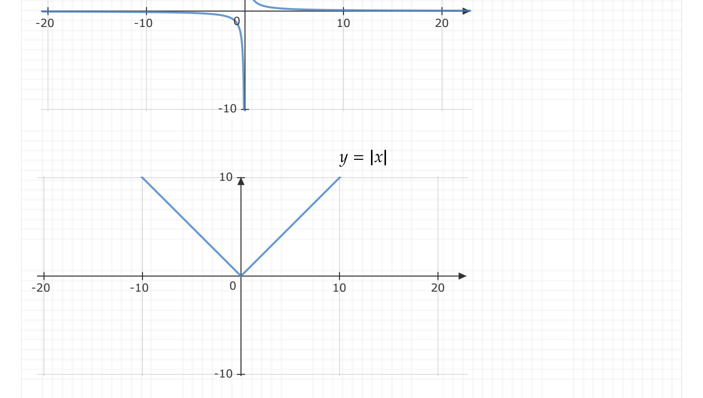


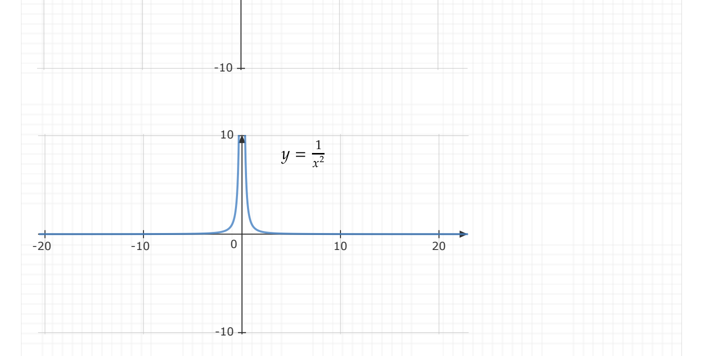


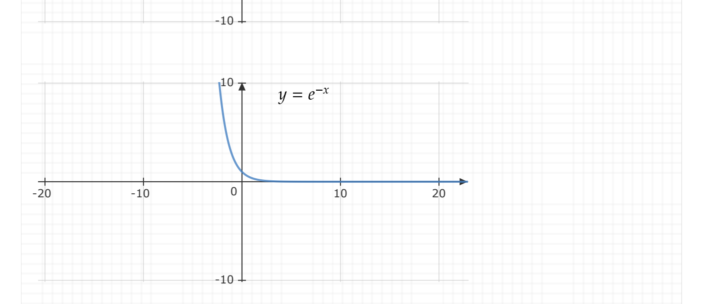


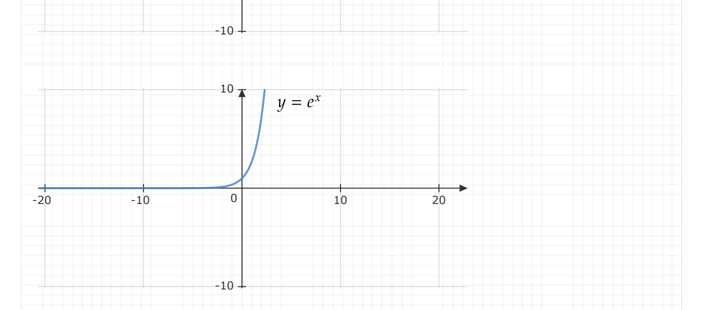


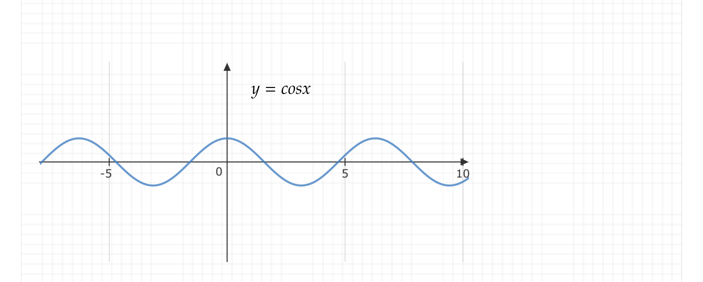


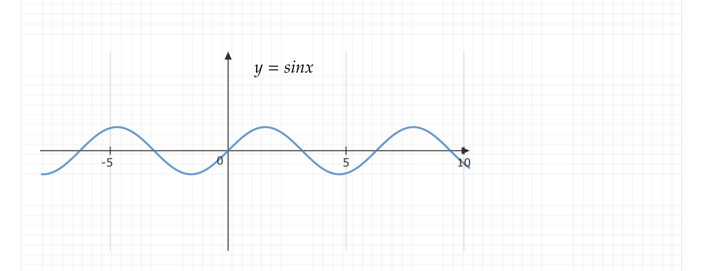


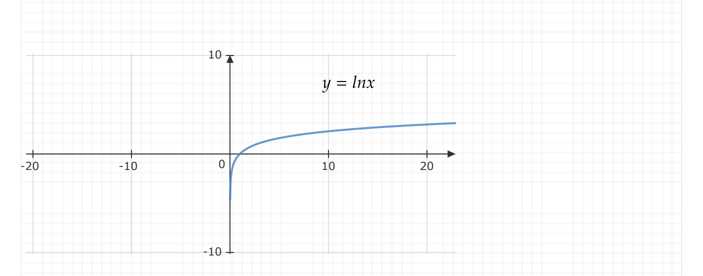


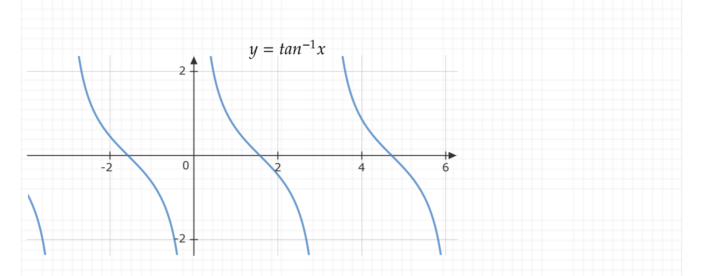


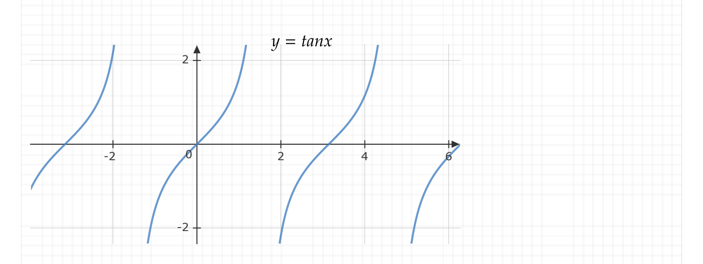


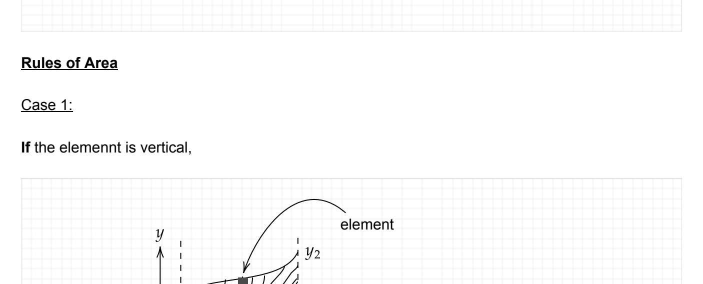


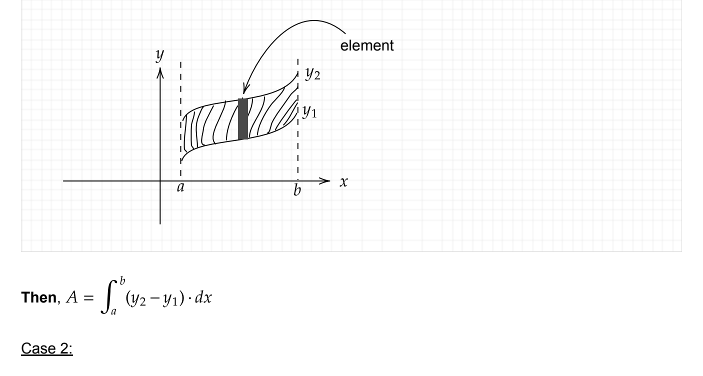








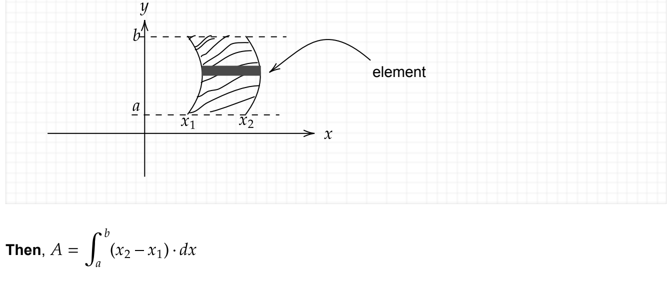




Rules of Area

Case 1:

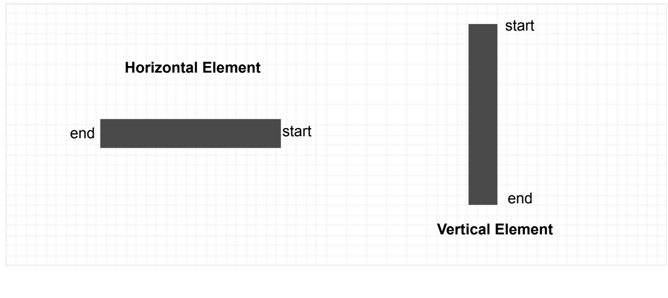
If the elemennt is vertical,



Then, $A = \int_a^b (y_2 - y_1) \cdot dx$

Case 2:

If the elemennt is horizontal,



Then, $A = \int_a^b (x_2 - x_1) \cdot dx$

Steps of solution:

1. Draw the area required (graphical representation)
2. find the points of intersection
3. use the correct element (horizontal or vertical)
4. Apply the rules and principles