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Preliminaries

- The purpose of utilizing the methods of solving ODE's is in order to find the output $x(t)$
- $\dot{x} = \dfrac{dx}{dt}$

Identifying the ODE:

Let these two equations be a general form of a second-order linear ordinary differential equation:

$$a\ddot{x} + b\dot{x} + cx = f(t) \tag{1}$$

$$a\ddot{x} + b\dot{x} + cx = 0 \tag{2}$$

Order: When identifying the order of an ordinary differential equation, the highest derivative determins the order.

Homogeneous: the equation is said to be homogeneous if the right-hand side is zero.

Non-homogeneous: the equation is said to be non-homogeneous if the right-hand side is non-zero.

Constant coefficients differential equation: the ODE is said to be a constant coefficients differential equation when a, b, and c are constants.

Linear differential equation: the ODE is said to be a linear differential equation when a, b, and c do not have non-linear coefficients.

Complex Numbers

Complex numbers are linear combinations of real and imaginary parts.

It is important to note that throughout this text, j will be used in place of i to represent,

$$j = \sqrt{-1}$$

This is done so as to eliminate confusion in the coming topics, namely in electrical systems when the concept of current is introduced.

Complex numbers can be represented in two forms,

Rectangular form:

$$z = x \pm jy$$

$$where,\ x\ is\ the\ real\ part\ \&\ jy\ is\ the\ imaginary\ part$$

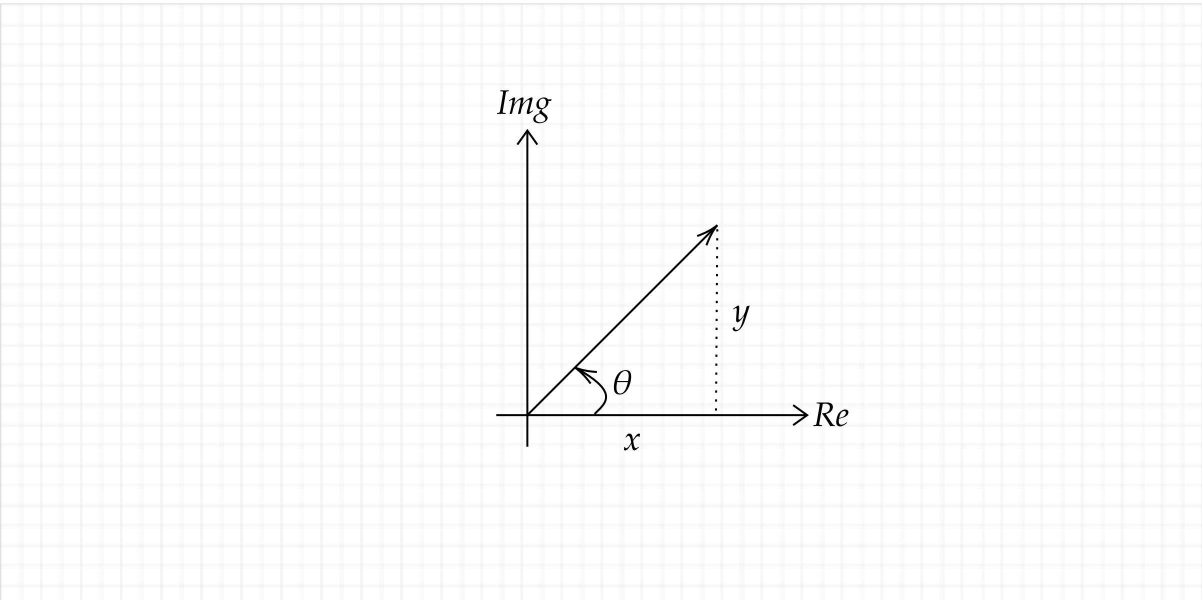
Polar form:

$$z = Ae^{\pm j\theta}$$

$$where,\ \ A = \sqrt{x^2 + y^2} \qquad (magnitude)$$

$$\&$$

$$\theta = \tan^{-1}(y/x) \quad (phase\ angle)$$



Euler's relation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

The left-hand side is called the harmonic form.

Suppose,

$$e^{a+jb}$$

Then using Euler's relation,

$$e^{a+jb} = e^a e^{jb} = e^a(\cos b + j\sin b)$$

Complex Number Operations:

$$let \quad z_1 = x_1 + jy_1 \;\; \& \;\; z_2 = x_2 + jy_2$$

Then,

- Squaring: $j^2 = -1$
- Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
- Subtraction: $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$
- Multiplication: $z_1 \cdot z_2 = x_1x_2 + jy_2x_1 + jy_1x_2 - y_1y_2)$

Complex Conjugate:

$$let \quad z = x + jy \;\; \& \;\; \overline{z} = x - jy$$

Then,

$$z\overline{z} = x^2 + xjy - xjy - y^2$$

$$z\overline{z} = x^2 - y^2$$

Some Properties:

- $z^n = \left\{Ae^{j\theta}\right\}^n = A^n e^{jn\theta} = A^n\{\cos\theta + j\sin\theta\}$

- $\dfrac{1}{j} = -j$

- $\dfrac{1}{z} = \dfrac{\overline{z}}{x^2+y^2}$

ODE's Solution Methods:

In this text, only two main solution methods will be explored.

They are:

1. Trial method/Undetermined coefficient method
2. Laplace transform method

<u>Method 1: Trial method/Undetermined coefficient method</u>

This method essentially converts the differential equation into an algebraic equation.

These are the steps to be followed:

<u>Step 1:</u>

$$let,$$

$$\begin{aligned} x(t) &= e^{mt} \\ \dot{x}(t) &= me^{mt} \\ \ddot{x}(t) &= m^2e^{mt} \end{aligned}$$

$$\vdots$$

$$etc.$$

<u>Step 2:</u> then substitute the above back into the original ODE

<u>Step 3:</u> find the characteristic equation, this is done by:

$$let,$$

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$then\ the\ characteristic\ equation\ becomes,$$

$$mx^2 + cx + k = 0$$

$$where,\ m_1\ \&\ m_2\ are\ roots$$

Step 4: the roots of the characteristic equation indicate the solution type, which in turn indicates the homogenous solution, x_h. If the ODE was homogeneous, then this is the only solution $x(t) = x_h$. Otherwise, if the ODE was non-homogeneous, then the particular solution, x_p, must also be obtained.

Types of x_h solutions based on roots cases:

As mentioned previously, the roots $m_1\ \&\ m_2$ determine the type of solution form, while the initial conditions determine $C_1\ \&\ C_2$.

Note:

- for function of time, I.C. (Initial Conditions)
- for function of not time, B.C. (Boundary Conditions)

Case 1: Distinct Real Roots (m_1, m_2)

$$x(t) = C_1e^{m_1t} + C_2e^{m_2t}$$

Case 2: Repeated Real Roots $(m_1, m_2,\ m_1 = m_2 = m)$

$$x(t) = C_1e^{mt} + C_2te^{mt}$$

Case 3: Imaginary Roots $(m_1, m_2 = \pm j\omega)$

$$x(t) = C_1 \sin \omega t + C_2 \cos \omega t$$

Case 3: Complex Conjugate Roots $(m_1 = a + j\omega,\ m_2 = a - j\omega)$

$$x(t) = e^{at}\{C_1 \sin \omega t + C_2 \cos \omega t\}$$

Finding the x_p solutions of non-homogeneous ODE:

Recall, that a non-homogeneous ODE takes the following form:

$$a\ddot{x} + b\dot{x} + cx = f(t)$$

$$where,\ f(t) \neq 0$$

The afforementioned steps helped determine the homogenous solution. In this subsection, the steps outlined here will help determine the particular solution.

$$x(t) \ = \ \underbrace{x_h(t)}_{\substack{homogeneous \\ solution}} \ + \ \underbrace{x_p(t)}_{\substack{particular \\ solution}}$$

The form of the particular solution can be determined by comparing $f(t)$ or parts of it to the table below. If more than one part of $f(t)$ is observed within the table below, superposition can be used.

$f(t)$	$x_p(t)$
$Constant$	$A = Constant$
t	$At + B$
t^2	$At^2 + Bt + C$
$\sin t$	$A \sin t + B \cos t$
$\cos t$	$A \sin t + B \cos t$
e^{at}	Ae^{at}

Outline of the steps:

1. Find x_h (using the steps in the previous subsection)
2. Find $x_p,\ \dot{x}_p,\ \ddot{x}_p$ (x_p is found using the table above, the rest are found by differentiating it w.r.t. time)
3. Substitute $x_p,\ \dot{x}_p,\ \ddot{x}_p$ in the original ODE
4. Compare c oefficients & solve for x_p's coefficients (If an element of x_p is similar to one in x_h then multiply it by t)
5. Make the general equation $x(t)$, then solve for intitial conditions to find the coefficeints C_1 and C_2

Laplace Transform Method

When trying to solve more complicated and lengthier ODE's, the Laplace transform method becomes of use. Especially when it comes to dealing with non-homogeneous ODE's and sets of ODE's.

The laplace method is simply a method of converting from time-dependent functions $f(t)$ from the time domain to the frequency domain $F(s)$. It effectively transforms a differential equation into an algebraic equation.

The definition of a laplace transform is as follows:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)\, e^{-st} \cdot dt \tag{3}$$

It will be used to generate the laplace transform of common functions:

<u>Laplace of a constant c</u>

$$\mathcal{L}\{c\} = \int_0^{\infty} c\, e^{-st} \cdot dt = c\, \frac{1}{s}\, e^{-st} \Big|_0^{\infty}$$

$$\mathcal{L}\{c\} = \frac{c}{s} \left\{e^{-\infty} - e^{0}\right\} = \frac{c}{s}\, \{0 - 1\}$$

$$\boxed{\mathcal{L}\{c\} = \frac{c}{s}}$$

<u>Laplace of t</u>

$$\mathcal{L}\{t\} = \int_0^{\infty} t\, e^{-st} \cdot dt$$

$let,$

$$t = u \qquad\qquad du = 1$$

$$e^{-st} = dv \qquad\qquad -\frac{1}{s}e^{-st} = v$$

by parts,

$$\mathcal{L}\{t\} = \left. uv \right|_0^\infty - \int v\, du$$

$$\mathcal{L}\{t\} = \left. \frac{-te^{-st}}{s} \right|_0^\infty + \int_0^\infty \frac{1}{s}\, e^{-st} \cdot dt$$

$$\mathcal{L}\{t\} = -\frac{1}{s^2}\left\{e^{-\infty} - e^0\right\}$$

$$\boxed{\mathcal{L}\{s\} = \frac{1}{s^2}}$$

If the defenition of Laplace is consistently used with common functions, a table such as this one below can be generated with ready-to-use transforms:

$X(s)$	$x(t),\ t \geqslant 0$
1	$\delta(t),\ \textit{unit impulse}$
${}^{1}\!/_{s}$	$u_s(t),\ \textit{unit step}$
${}^{c}\!/_{s}$	$\textit{constant } c$
$\dfrac{e^{-sD}}{s}$	$u_s(t-D),\ \textit{shifted unit step}$
$\dfrac{n!}{s^{n+1}}$	t^n
$\dfrac{1}{s+a}$	e^{-at}
$\dfrac{1}{(s+a)^n}$	$\dfrac{1}{(n-1)!} \cdot t^{n-1}\, e^{-at}$
$\dfrac{b}{s^2+b^2}$	$\sin bt$
$\dfrac{s}{s^2+b^2}$	$\cos bt$
$\dfrac{b}{(s+a)^2+b^2}$	$e^{-at} \cdot \sin bt$
$\dfrac{s+a}{(s+a)^2+b^2}$	$e^{-at} \cdot \cos bt$

Tabulated below are the properties of the laplace transform:

Properties of the Laplace Transform	
$\mathcal{L}\{af(t)\}$	$a\,F(s)$
$\mathcal{L}\{ax+by\}$	$aX(s)+bY(s)$
$\mathcal{L}\{tf(t)\}$	$-\frac{d}{ds}\,F(s)$
$\mathcal{L}\left\{t^n f(t)\right\}$	$(-1)^n\,\frac{d^n}{ds^n}\,F(s)$
$\mathcal{L}\{\dot{x}\}$	$sX(s)-x(0)$
$\mathcal{L}\{\ddot{x}\}$	$s^2X(s)-sx(0)-\dot{x}(0)$
$\mathcal{L}\{\dddot{x}\}$	$s^3X(s)-s^2x(0)-s\dot{x}(0)-\ddot{x}(0)$
$\mathcal{L}\left\{e^{-at}f(t)\right\}$	$F(s+a)$

Note:

- $x(0), \dot{x}(0),\ and\ \ddot{x}(0)$ are initial conditions
- a shift in the time domain introduces an exponential in the frequency domain
- a shift in the frequency domain introduces an exponential in the time domain

Common Functions

Below are some common functions and their laplace transforms.

1) Ramp Function

$$R(t) = \begin{cases} 0 & for\ t < 0 \\ Rt & for\ t \geqslant 0 \end{cases}$$

$$\mathfrak{L}\{R(t)\} = \frac{R}{s^2}$$

2) Step Function

$$Mu_s(t) = \begin{cases} 0 & ,\ for\ t < 0 \\ M & ,\ for\ t \geqslant 0 \end{cases}$$

$$\mathfrak{L}\{Mu_s(t)\} = \frac{M}{s}$$

$$N.b.\ \ if\ M = 1,\ then\ it\ is\ a\ unit\ step\ function.$$

3) Shifted Step Function

$$Mu_s(t-D) = Mu_D(t) = \begin{cases} 0 & ,\ for\ t < D \\ M & ,\ for\ t \geqslant D \end{cases}$$

$$\mathfrak{L}\{Mu_s(t-D)\} = \frac{M}{s}\,e^{-sD}$$

4) Pulse Function

$$P(t) = \begin{cases} M & ,\ 0 < t < D \\ 0 & ,\ elsewhere \end{cases}$$

$$\mathfrak{L}\{P(t)\} = \frac{M}{s}\left(1 - e^{-sD}\right)$$

The above laplace transform for the pulse function can be shown to be the case by the following,

$$\mathfrak{L}\{Mu_s(t)\} - \mathfrak{L}\{Mu_s(t-D)\} = \mathfrak{L}\{P(t)$$

$$\frac{M}{s} - \frac{M}{s}\,e^{-sD} = \frac{M}{s}\left(1 - e^{-sD}\right)$$

5) Unit-Impulse Function (Dirac Delta $\delta(t)$)

$$f(t) = \frac{1}{D},\ F(s) = \lim_{D \to 0}\left(\frac{se^{-s\tau}}{s}\right) = \lim_{D \to 0}\left(\frac{se^{-s\tau}}{s}\right) = \lim_{D \to 0}\left(e^{-s\tau}\right) = 1$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-sa}$$

$$eg.\ \ \mathcal{L}\{\delta(t)\} = 1\ \&\ \mathcal{L}\{5\ \delta(t)\} = 5,$$
$$where\ 5\ represents\ the\ intensity.$$

Laplace Inverse

The laplace inverse is performed with the objective to convert from the frequency domain to the time domain using the laplace inverse property.

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = f(t)}$$

$$Y(s) = \frac{C(s)}{D(s)} = \frac{c_m S^m + c_{m-1} S^{m-1} + \ldots + c_0}{d_n S^n + d_{n-1} S^{n-1} + \ldots + d_0}$$