Abdulaziz Al Ghannami
www.ghannami.com

Spring & Damper Elements in Mechanical Systems

Spring/Elastic Element

Resisting force is a function of displacement.

- $L \implies$ Free-length with compressive/tensile forces applied to it.
- Linear force of deflection model:

$$f = k\,x,\ where\ k\ is\ the\ spring\ constant$$
$$\&$$
$$x\ is\ the\ compression/extention$$

For a spring with a round wire,

$$k = \frac{Gd^4}{64nR^3},\ where\ G\ is\ the\ shear\ modulus\ \&\ n\ is\ the\ \#\ of\ coils$$

For to determine the force using the flexure formula:

$$F = \frac{AE}{L}\cdot x,\ note\ that\ \frac{AE}{L} = k$$

- Beams with large forces applied to them act like springs $\implies$ used to find k by knowing beam geometry, beam material, method of support

Torsional Spring Element

Equation is comparable with that of the translational spring element:

$$T = k_T\theta,\ where\ k_T\ is\ the\ torsional\ spring\ constant\ (FPS:\ lbft/rad,\ SI:\ N\cdot m/rad)$$
$$\&$$
$$\theta\ is\ the\ net\ angular\ twist\ of\ element$$

- k_T: depends on the geometry of element and material properties (E/G)
- For solid cylinder:

$$k_T = \frac{\pi D^4}{32\,L},\ if\ hollow\ use\ \left(D^4 - d^4\right)\ in\ place\ of\ D^4,$$

$$where\ D\ is\ the\ outer\ diameter\ \&\ d\ is\ the\ inner\ diameter$$

- For round coil spring:

$$k_T = \frac{Ed^4}{64nD}$$

<u>Series & Parallel Spring Elements</u>

Parallel (both nodes start and end at same points) $\Longrightarrow$ same x or θ $\longrightarrow k_e = \sum_{i=1}^{n} k_i$

Series (springs connected node to node) $\Longrightarrow$ same f or T $\longrightarrow \frac{1}{k_e} = \sum_{i=1}^{n} \frac{1}{k_i}$

N.b. for series the k equivalent is smaller than the individual k's

Assumptions:

- Mass of is very small compared to other masses in the system
- System is in static equilibrium

<u>Modelling Mass-Spring Systems</u>

- The point mass assumption: mass of object is concentrated at the center
- if not explicitly stated, must assume that the spring mass can be neglected
- Ideal spring element is massless
- a real spring can be represented as an ideal spring by:
  - (a) Neglecting its mass
  - (b) including it in another mass in the system
- if the spring is at free length and object is inclined or vertical, then one must take into consideration gravity
- However, if the spring is at static equilibrium δ_{st} then $\sum F = 0$ and it is at static equilibrium so thhe effect of gravity has taken plaxe and need not be accounted for.
- Force due to gravity is canceled out, as the displacement of thr mass is measured from the equilibrium position.
- Static Spring Force $\Longrightarrow$ spring force caused by its static deflection δ_{st}
- Dynamic Spring Force $\Longrightarrow$ spring force caused by the variable deflection δ_{st}
- Advantages of choosing the equilibrium position as the coordinate origin:

    - No need to specify the geometry dimension of the mass
    - this choice simplifies the equation of motion by eliminating hte static forces
- For small angles, $r \sin\theta = r\theta$ etc.

Solving the Equation of Motion (EOM)

Suppose the following form, a mass and spring system with an input force f:

$$m\ddot{x} + kx = f,$$

$$\textit{where } f \textit{ is the applied force other than gravity \& spring force}$$

Let $f = 0$, let mass in motion be pulled to the position $x(0)$, initial velocitiy $\dot{x}(0)$

Then,

$$\implies x(t) = c_1 \sin\omega_n t + c_2 \cos\omega_n t$$

$$\implies \omega_n = \sqrt{\frac{k}{m}}$$

$$\implies T = \frac{2\pi}{\omega_n}, \textit{ where } T \textit{ is the period}$$

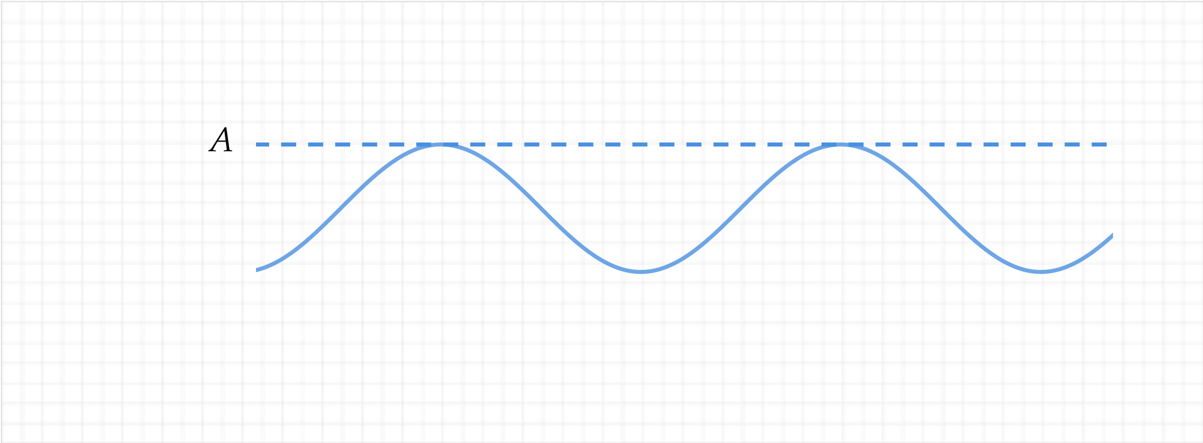
$$\implies c_1 = \dot{x}(0)/\omega_n, \;\; c_2 = x(0)$$

$$\implies x(t) = A\sin(\omega t + \phi)$$

where $\sin(\phi) = \dfrac{x(0)}{A}$ & $\cos(\phi) = \dfrac{\dot{x}(0)}{A\omega_n}$

$$\therefore\; A = \sqrt{[x(0)]^2 + \left[\frac{\dot{x}(0)}{\omega_n}\right]^2}$$

- Simple Harmonic Motion: a type of motion where acceleration is proportional to displacement but opposite in sign.



Equivalent Mass of Elastic Element

- if the mass of the elastic element (eg. spring) is to be considered, then one must first the equivalent mass. It is computed using kinetic energy (K.E.) equivalence.
- Mass equivalent of Helical spring with rod in extension/compression

$$m_e = \frac{m_r}{3} \; or \; \frac{m_s}{3}$$

Energy Methods

Potential Energy (P.E.) of a spring $\implies$ Translational: $V(x) = \frac{1}{2} k\, x^2$,

Torsional: $V(\theta) = \frac{1}{2} k_T\, \theta^2$

Conservation of Energy:

$$T + V = T_o + V_o = Constant$$

$$or$$

$$\Delta T + \Delta V = 0$$

If gravity is taken into account,

$$T + V_g + V_s = Constant$$

$$or$$

$$\Delta T + \Delta V_g + \Delta V_s = Constant$$

N.b. T is the Kinetic Energy (K.E.)

If $T + V$ is differentiated with respect to time then the EOM is obtained.

<u>Dampers/Damping Elements</u>

The resisting force is a function of velocity.

- A damping element is an element that resists the relative velocity across it.
- The resisting force in a damper is caused by fluid friction.
- The force depends on the relative velocity of the piston & the cylinder so the faster the piston is moved relative to the cylinder, the greater thhe resisting force.
- Dampers do not exert forces unless its endpoints are moving relative to eachother.
- A rotary or torsional damper exerts a resisting torque in response to an angular velocity difference across it.

<u>Ideal Dampers</u>

- An ideal damping element is one that is massless.
- Unless explicitly stated, assume damper mass can be neglected.
- If piston mass & cylinder mass are substituted, the damper must be modeled as two masses: one for the piston and one for the cylinder.

<u>Damper Representation</u>

- Translational damper:

$$f = cv = c\dot{x},\ where\ c\ is\ the\ damping\ coefficient$$

- The damping coefficient of a piston-type damper with single hole:

$$c = 8\pi\mu L\left[\left(\frac{D}{d}\right)^2 - 1\right]^2$$

- Torsional damper:

$$T = c_T \omega = c\dot{\theta},\ where\ c_T\ is\ the\ torsional\ damping\ coefficient$$

- The damping coefficient of a journal bearing using Petrov's law for a simple bearing:

$$c_T = \frac{\pi D^3 L \mu}{4\epsilon},\ where\ \epsilon\ is\ the\ thickness\ if\ the\ lubricating\ layer$$

- Finding the free response gives: $x(t),\ \zeta,\ \tau$
- For coupled spring and damper in order to obtain only one equation by placing the two into one, perform laplace transform and substitute one equation in another. However, doing so will increase the order of the equation by one.
- Time constant (τ) is the reciprocal of the real part of the roots. Recall that 4τ is how long it takes for the system to become stable.
- The dominant τ is from the largest reciprocal of the real part of the roots and that same root defines the response eg. oscillatory etc.

$$if\ \frac{2\pi}{\omega_n} = T > 4\tau$$

$$then\ complete\ oscillation\ will\ not\ be\ observed\ in\ the\ response.$$

- Suppose the EOM is of the form: $\dot{x} + ax = b$

$$x(t) = \frac{b}{a} + \left[x(0) - \frac{b}{a}\right] \cdot e^{-at}$$

- System is stable if a & b are positive: $\ddot{\phi} + a\,\dot{\phi} + b\,\phi = 0$

<u>Lagrangian Mechanics</u>

This methodology is also used to determine the EOM using energy.

Finding the Lagrangian:

$$\mathscr{L} = T - V$$

$$where\ \mathscr{L}\ is\ the\ lagrangian,\ T\ is\ the\ kinetic\ energy,\ and\ V\ is\ the\ potential\ energy$$

Construct the following equation (this equation results in the EOM):

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q_i}}\right) - \frac{\partial \mathscr{L}}{\partial q_i} = Q_{qi} \tag{1}$$

Steps:

1. Define $T,\ V,\ and\ \mathscr{L}$
2. Find partial work δW and from partial work find the coefficients of $\delta x \Longrightarrow$ to find Q_x. Dampers are accounted for here because they dissipate work so are negative.
3. Then substitute into equation (1)
4. If there is more than one mass, then solve for q_1 and q_2 etc.

Note that x can be θ.